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On the Completeness and Other Properties of Some Function Systems in L_p, $0 < p < \infty$

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We study minimal conditions under which a function system is a representation system in $L_p(a, b)$, $-\infty \leqslant a < b \leqslant +\infty$, $0 < p < \infty$, i.e., any function $f \in L_p(a, b)$ can be represented by at least one L_p-convergent series relative to this system.

The interest in function systems of the type

$$\{\varphi_{n,k}\}_{n,k \in Z} = \{\varphi(a^n t - kb)\}_{n,k \in Z},$$

where $a > 1$, $b > 0$, and φ is an arbitrary function from L_p, arose in connection with investigations of wavelets [BDR, D, JL, JM, Ml, Md, Mey] and frames [ChSh1, ChSh2] and in connection with questions of image compression. The questions of completeness of integer translates in function spaces on R are considered in [AO]. Moreover, the authors cited above and others continue to attempt to generalize the classical systems (Haar, trigonometric, Faber–Schauder, etc.) to get systems with new properties.

Since there exist an infinite number of function systems, the question arises: "How can we find the optimal function system for a given problem?" To answer such a question we should somehow classify all such function systems.

Properties of function systems depend on the function spaces (for example; there is no basis in $L_p(0, 1)$, $0 < p < 1$). Therefore, it is natural to consider function systems in the spaces: (1) $L_p(0, 1)$, $0 < p < 1$, (2) $L_p(0, 1)$, $1 \leqslant p < \infty$ (if the field of investigation is limited by the scale of L_p, $p > 0$).

The notion of a representation system which generalizes the notion of a basis was introduced by A. A. Talalyan [T1]. There are investigations on subsystems of representation systems [I, F1, F2], on representation systems in

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$\Phi(L)$ [U, I, O1, O2, F1, F2], and on the representation of complex functions by series of exponentials [K], etc.

DEFINITION 1 [T1, T2]. A system $\{f_n\}_{n=1}^{\infty} \subset L_p$, $0<p<\infty$, is called a representation system in the space L_p if for any $f \in L_p$ there exists a series $\sum_{k=1}^{\infty} c_k f_k$ such that

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} c_k f_k \right\|_p = 0.$$

Here $\|f\|_p = (\int_a^b |f(t)|^p\, dt)^{\min(1,1/p)}$, $0<p<\infty$, $-\infty \leqslant a < b \leqslant +\infty$. This definition generalizes to F-spaces.

DEFINITION 2. A system $\{f_n\}_{n=1}^{\infty} \subset L_p$, $0<p<\infty$, is called a complete system in the space L_p if for any $f \in L_p$ and for any $\varepsilon>0$ there exists a finite sum $\sum_{k=1}^{k_0} c_k f_k$ such that

$$\left\| f - \sum_{k=1}^{k_0} c_k f_k \right\|_p < \varepsilon.$$

We can remark that each basis is a representation system, but not vice versa and each representation system is a compete system, but not vice versa.

The results below are more general than the results in [F3, FO].

LEMMA 1. *Let $\varphi \in L_p(a,b)$, $-\infty \leqslant a < b \leqslant +\infty$, $1 \leqslant p < \infty$, $\int_a^b \varphi(t)\, dt = \delta \neq 0$. There exist constants λ, l_1, $l_2 \in R$, $\lambda \neq 0$; $a \leqslant l_1 < l_2 \leqslant b$ such that*

$$\sigma \equiv \left(\frac{1}{l_2 - l_1}\right)^{1/p} \|\chi_{(l_1,l_2)}(t) - \lambda\varphi(t)\|_p < 1. \tag{1}$$

Here $\chi_{(c,d)}(t)$ denotes the characteristic function of (c,d).

Proof. Let for simplicity $0<\delta<1$ and $\lambda>0$ (if $\delta<0$ then consider $-\varphi$). For $p \geqslant 1$ we have the inequalities

$$|1-x|^p \leqslant 1 - px + c_0 x^2, \qquad |x| \leqslant \tfrac{1}{2},$$

$$|1-x|^p \leqslant 1 + c_1|x| + c_2|x|^p, \qquad x \in \mathbf{R},$$

with some positive constants c_0, c_1, c_2. Let

$$F_{\alpha,l_1,l_2} = \left\{ l_1 < t < l_2 \colon |\varphi(t)| \leqslant \frac{1}{2\alpha} \right\},$$

$$F^*_{\alpha,l_1,l_2} = \left\{ l_1 < t < l_2 \colon |\varphi(t)| > \frac{1}{2\alpha} \right\}, \qquad \alpha > 0.$$

Obviously, there exist l_1^0, l_2^0, α_0 such that if $a \leqslant l_1 \leqslant l_1^0 < l_2^0 \leqslant l_2 \leqslant b$, $\alpha_0 > \alpha > 0$, then

$$\int_{F_{\alpha, l_1, l_2}} \varphi(t)\, dt > \tfrac{7}{8}\delta, \qquad \int_{(a, b)\setminus(l_1, l_2)} |\varphi(t)|^p\, dt < \tfrac{1}{8}\delta,$$

$$c_1 \int_{F^*_{\alpha, l_1, l_2}} |\varphi(t)|\, dt < \tfrac{1}{8}\delta, \qquad c_2 \int_{F^*_{\alpha, l_1, l_2}} |\varphi(t)|^p\, dt < \tfrac{1}{8}\delta.$$

If we take λ, $0 < \lambda < \alpha < \alpha_0 < 1$, such that $c_0\lambda/4\alpha^2 < \delta/8(l_2 - l_1)$ and $p\lambda < l_2 - l_1$, then

$$\begin{aligned}
&\frac{1}{l_2 - l_1} \|\chi_{(l_1, l_2)}(t) - \lambda\varphi(t)\|_p^p \\
&\quad = \frac{1}{l_2 - l_1}\left(\int_{F_{\alpha, l_1, l_2}} |1 - \lambda\varphi(t)|^p\, dt + \int_{F^*_{\alpha, l_1, l_2}} |1 - \lambda\varphi(t)|^p\, dt \right.\\
&\qquad \left. + \lambda^p \int_{(a, b)\setminus(l_1, l_2)} |\varphi(t)|^p\, dt\right) \\
&\quad \leqslant \frac{1}{l_2 - l_1}\left((l_2 - l_1) - p\lambda \int_{F_{\alpha, l_1, l_2}} \varphi(t)\, dt + c_1\lambda \int_{F^*_{\alpha, l_1, l_2}} |\varphi(t)|\, dt \right.\\
&\qquad \left. + c_2\lambda^p \int_{F^*_{\alpha, l_1, l_2}} |\varphi(t)|^p\, dt + \frac{c_0\lambda^2(l_2 - l_1)}{4\alpha^2} + \lambda^p \int_{(a, b)\setminus(l_1, l_2)} |\varphi(t)|^p\, dt\right) \\
&\quad \leqslant 1 - \frac{3p\lambda}{8(l_2 - l_1)}\delta < 1. \quad \blacksquare
\end{aligned}$$

Lemma 2. *Let* $\varphi \in L_2(a, b) \cap L_p(a, b)$, $-\infty \leqslant a < b \leqslant +\infty$, $0 < p < 1$, $\int_a^b |\varphi(t)|^2\, dt = \delta \neq 0$. *Then there exist constants* $\lambda, l_1, l_2 \in R$, $\lambda \neq 0$; $a \leqslant l_1 < l_2 \leqslant b$ *such that*

$$\sigma \equiv \frac{1}{l_2 - l_1} \|\chi_{(l_1, l_2)}(t) - \lambda\varphi(t)\|_p < 1, \qquad 0 < p < 1.$$

Here $\chi_{(c, d)}(t)$ *denotes the characteristic function of* (c, d).

Proof. If $0 < p < 1$, then we easily obtain by the Taylor formula that

$$|1 - x|^p \leqslant 1 - px - cx^2, \qquad |x| \leqslant \tfrac{1}{2},$$

where c is some positive constant. Let

$$P(\lambda)=\frac{1}{l_2-l_1}\int_a^b |\chi_{(l_1,\,l_2)}(t)-\lambda\cdot\varphi(t)|^p\,dt, \qquad \lambda\in R,$$

$$F_{\lambda,\,l_1,\,l_2}=\left\{t\colon l_1<t<l_2,\ |\lambda\varphi(t)|\leqslant\frac{1}{2}\right\}=F_{-\lambda,\,l_1,\,l_2},$$

$$F^*_{\lambda,\,l_1,\,l_2}=\left\{t\colon l_1<t<l_2,\ |\lambda\varphi(t)|>\frac{1}{2}\right\}=F^*_{-\lambda,\,l_1,\,l_2}.$$

Then we obtain

$$\begin{aligned}P(\lambda)=\int_{F_{\lambda,\,l_1,\,l_2}}+\int_{F^*_{\lambda,\,l_1,\,l_2}}+\int_{(a,\,b)\setminus(l_1,\,l_2)}&\leqslant 1-\frac{p\lambda}{l_2-l_1}\int_{F_{\lambda,\,l_1,\,l_2}}\varphi(t)\,dt\\&-\frac{c}{l_2-l_1}\int_{F_{\lambda,\,l_1,\,l_2}}|\lambda\varphi(t)|^2\,dt+\frac{1}{l_2-l_1}\int_{F^*_{\lambda,\,l_1,\,l_2}}|\lambda\varphi(t)|^p\,dt\\&+\frac{1}{l_2-l_1}\int_{(a,\,b)\setminus(l_1,\,l_2)}|\lambda\varphi(t)|^p\,dt.\end{aligned}$$

If $t\in F^*_{\lambda,\,l_1,\,l_2}$, then we have $|\lambda\varphi(t)|^p\leqslant 2^{2-p}\,|\lambda\varphi(t)|^2=2^{2-p}\,|\lambda|^2|\,\varphi(t)|^2$. It is obvious that there exist λ_0, l_1^0, $l_2^0\in R$, $l_1^0<l_2^0$ such that

$$l_2^0-l_1^0>|\lambda_0|^2\,c, \qquad \int_{F_{\lambda,\,l_1,\,l_2}}|\varphi(t)|^2\,dt>\frac{7}{8}\,\delta, \qquad \frac{2^{2-p}}{c}\int_{F^*_{\lambda,\,l_1,\,l_2}}|\varphi(t)|^2\,dt<\frac{1}{8}\,\delta$$

for all $|\lambda|\leqslant|\lambda_0|<1$ and $l_1\leqslant l_1^0$, $l_2\geqslant l_2^0$. Take $\lambda=\lambda_0$ and $l_1^1\leqslant l_1^0$, $l_2^1\geqslant l_2^0$ such that the following inequality holds:

$$|\lambda_0|^p\int_{(a,\,b)\setminus(l_1^1\,l_2^1)}|\varphi(t)|^p\,dt<|\lambda_0|^2\,c\,\tfrac{2}{4}\delta.$$

Thus we get

$$\begin{aligned}\frac{P(\lambda_0)+P(-\lambda_0)}{2}&\leqslant 1+\frac{1}{l_2^1-l_2^1}\Bigg(|\lambda_0|^p\int_{(a,\,b)\setminus(l_1^1\,l_2^1)}|\varphi(t)|^p\,dt\\&\qquad-|\lambda_0|^2\left[c\int_{F_{\lambda_0,\,l_1^1,\,l_2^1}}|\varphi(t)|^2\,dt-2^{2-p}\int_{F^*_{\lambda_0,\,l_1^1,\,l_2^1}}|\varphi(t)|^2\,dt\right]\Bigg)\\&=\sigma<1. \quad \blacksquare\end{aligned}$$

From Lemmas 1, 2 we have that there exist $\lambda \in R$ and a measurable set $Q \subset (a, b)$, $0 < |Q| < \infty$, such that

$$\frac{1}{|Q|^{c/p}} \|\chi_Q(t) - \lambda \cdot \varphi(t)\|_p = \sigma < 1,$$

where $c = 1$ if $p \geqslant 1$ and $c = p$ if $0 < p < 1$. Consider the function systems $\{\varphi_n(t)\}_{n \in N}$ such that

$$\sup_n \sigma_n = \sigma < 1, \tag{2}$$

where $\sigma_n = \inf \{\lambda \in R, Q \subset (a, b): (1/|Q|^{c/p}) \|\chi_Q(t) - \lambda \varphi_n(t)\|_p\}$. It is obvious that if $p \geqslant 1$ and $\sup_n \sigma_n = \sigma < 1$, then $\int_a^b \varphi_n(t)\, dt \neq 0$. If $\varepsilon > 0$ is such that $\sigma + \varepsilon = \sigma' < 1$, then there exist $\lambda_n \in R$, $Q_n = \bigcup_{i=1}^{i_n} [a_i^n, b_i^n]$, such that

$$\sigma'_n = \frac{1}{|Q_n|^{c/p}} \|\chi_{Q_n}(t) - \lambda_n \cdot \varphi_n(t)\|_p \leqslant \sigma + \varepsilon = \sigma' < 1 \tag{3}$$

and also $\sup_n \sigma'_n \leqslant \sigma' < 1$. Let the system $\{\varphi_n\}_{n \in N}$ satisfy the condition

$$\forall N \in \mathbf{N}, \qquad \text{mes} \left\{ (a, b) \Big\backslash \bigcup_{n=N}^{\infty} Q_n \right\} = 0. \tag{4}$$

Here and below, we will use the term “mes” to signify Lebesgue’s measure.

Let $x_n = \min_i \{a_i^n\}$, $y_n = \max_i \{b_i^n\}$, and denote $d(\varphi_n) = y_n - x_n$. Denote $\text{supp}\, \varphi_n = \{t: \varphi_n(t) \neq 0\}$. Let

$$d(\varphi_n) \to 0, \qquad n \to \infty, \quad d(\varphi_n) \neq 0. \tag{5}$$

Let us call the function $\chi_{Q_n}(t)$ the main characterizing function of the element $\varphi_n(t)$ of the system $\{\varphi_n\}$. Let

$$A_n = \inf \{x \in (a, b): \forall \varepsilon > 0\ \text{mes}\{(x, x + \varepsilon) \cap \text{supp}\, \varphi_n\} \neq 0\},$$

$$B_n = \sup \{y \in (a, b): \forall \varepsilon > 0\ \text{mes}\{(y - \varepsilon, y) \cap \text{supp}\, \varphi_n\} \neq 0\}.$$

Denote $D_n = (A_n, B_n)$.

Below, we use Dunford and Schwartz’s definition [DSch, p. 30, 231] of Vitali’s covering.

THEOREM 1. *Assume that a subsystem* $\{\varphi_{n_k}\}$ *of the system* $\{\varphi_n\}_{n \in N} \subset L_1(a, b)$, $-\infty \leqslant a < b \leqslant +\infty$ *satisfies the properties* (3), (4), (5) *and for each* $N \in \mathbf{N}$ *the set* (a, b) *is covered in Vitali’s sense by the family* $\{Q_{n_k}\}_{k=N}^{\infty}$. *Then if* $N \in \mathbf{N}$ *the subsystem* $\{\varphi_{n_k}\}_{k=N}^{\infty}$ *is a representation system in* $L_1(a, b)$.

THEOREM 2. *Let for a subsystem $\{\varphi_{n_k}\}$ of the system $\{\varphi_n\}_{n\in N}\subset L_p(a,b)$, $-\infty\leqslant a<b\leqslant+\infty$, $1\leqslant p<\infty$ the following properties hold,*

$$|D_{n_k}|\to 0,\qquad k\to\infty,\qquad |D_{n_k}|\neq 0,$$

$$\forall N\in\mathbf{N},\qquad \operatorname{mes}\left\{(a,b)\Big\backslash\bigcup_{k=N}^{\infty} D_{n_k}\right\}=0,$$

$$\sup_k \sigma_{n_k}=\sigma<1,$$

where $\sigma_{n_k}=\inf\{\lambda\in R\colon (1/|D_{n_k}|^{1/p})\,\|\chi_{D_{n_k}}(t)-\lambda\varphi_{n_k}(t)\|_p\}$. *Then for arbitrary $N\in\mathbf{N}$ the subsystem $\{\varphi_{n_k}\}_{k=N}^{\infty}$ is a representation system in $L_p(a,b)$, $1\leqslant p<\infty$.*

It is obvious that the case $p=1$ in Theorem 2 is a special case of Theorem 1.

THEOREM 3. *Assume that a subsystem $\{\varphi_{n_k}\}$ of the system $\{\varphi_n\}_{n\in N}\subset L_2(a,b)\cap L_p(a,b)$, $-\infty\leqslant a<b\leqslant+\infty$, $0<p<1$, satisfies the properties (3), (4), (5) and for each $N\in\mathbf{N}$ the set (a,b) is covered in Vitali's sense by the family $\{Q_{n_k}\}_{k=N}^{\infty}$. Then for arbitrary $N\in\mathbf{N}$ the subsystem $\{\varphi_{n_k}\}_{k=N}^{\infty}$ is a representation system in $L_p(a,b)$, $0<p<1$.*

LEMMA 3. *Let for some subsystem $\{\varphi_{n_k}\}$ of the system $\{\varphi_n\}$ the conditions of Theorem 1 or 2, or 3 be fulfilled. Then for any step function $R(t)$ and arbitrary $N\in\mathbf{N}$, there exist a finite sum $P(t)\equiv\sum_{k=N}^{m}c_k\varphi_{n_k}$, $m>N$, and σ_0', $\sigma'<\sigma_0'<1$, such that*

$$\|R(t)-P(t)\|_p\leqslant\sigma_0'\,\|R(t)\|_p, \tag{6}$$

$$\left\|\sum_{k=N}^{n}c_k\varphi_{n_k}\right\|_p\leqslant 4\,\|R\|_p,\qquad N\leqslant n\leqslant m, \tag{7}$$

where $\sigma'<1$ is defined in (3) (note that in the case of Theorem 2, condition (3) is fulfilled too, and we put $Q_{n_k}=D_{n_k}$ everywhere below for this theorem).

Proof. For brevity, we will denote $p_k=\lambda_{n_k}\varphi_{n_k}$ where λ_{n_k} is taken from (3), and $\{\varphi_{n_k}\}_{k=1}^{\infty}$ is a subsystem of the system $\{\varphi_n\}$. Let

$$R(t)=\sum_{k=1}^{M}d_k\chi_{(\alpha_k,\beta_k)}(t)$$

be the given step function, with $\{(\alpha_k,\beta_k)\}$ the corresponding system of pairwise disjoint intervals from (a,b), and $\chi_{(c,d)}(t)$ denoting the characteristic function of (c,d). From Vitali's Theorem [DSch, p. 232] we can obtain that for each finite interval $E_k=(\alpha_k, \beta_k)$, $1\leqslant k\leqslant M$, arbitrary $\varepsilon>0$ there exist L and sets Q_{n_l} such that $\text{mes}\{E_k\setminus\bigcup_{l=1}^{L} Q_{n_l}\}<\varepsilon$, where $Q_{n_l}\subset E_k$, $Q_{n_l}\cap Q_{n_i}=\varnothing$ if $l\neq i$ (note that in the case of Theorem 2 the set (a, b) is covered in Vitali's sense by the family $\{D_{n_k}\}$). Take such sets Q_{n_l} that

$$\left\| R(t)-\sum_{k=1}^{M} d_k \sum_{l=r_k}^{l_k} \chi_{Q_{n_l}}(t)\right\|_p \leqslant(\sigma'_0-\sigma')\ \|R(t)\|_p.$$

Fix these numbers and elements. Then we obtain

$$\begin{aligned}
&\left\| R(t)-\sum_{k=1}^{M} d_k \sum_{l=r_k}^{l_k} p_l\right\|_p \\
&\quad\leqslant \left\| R(t)-\sum_{k=1}^{M} d_k \sum_{l=r_k}^{l_k} \chi_{Q_{n_l}}(t)\right\|_p \\
&\qquad+\left\| \sum_{k=1}^{M} d_k \sum_{l=r_k}^{l_k} \chi_{Q_{n_l}}(t)-\sum_{k=1}^{M} d_k \sum_{l=r_k}^{l_k} p_l(t)\right\|_p \\
&\quad\leqslant(\sigma'_0-\sigma')\ \|R(t)\|_p+\left(\sum_{k=1}^{M} |d_k|^p \sum_{l=r_k}^{l_k} (\sigma'_{n_l})^{p(1/c)}\ |Q_{n_l}|\right)^{(1/p)\,c} \\
&\quad\leqslant\sigma'_0\ \|R(t)\|_p.
\end{aligned}$$

We now prove (7). Let $N\leqslant n\leqslant m$. Then there exists an i for which $1\leqslant i\leqslant M$ such that $r_i\leqslant n\leqslant l_i$ or $l_i<n<r_{i+1}$ and

$$\begin{aligned}
\left\| \sum_{l=N}^{n} c_l p_l\right\|_p &= \left\| \sum_{k=1}^{i-1} d_k \sum_{l=r_k}^{l_k} p_l+\sum_{l=r_i}^{n} c_l p_l\right\|_p \\
&\leqslant \left\| \sum_{k=1}^{i-1} d_k \sum_{l=r_k}^{l_k} p_l-\sum_{k=1}^{i-1} d_k \sum_{l=r_k}^{l_k} \chi_{Q_{n_l}}\right\|_p \\
&\quad+\left\| d_i \sum_{l=r_i}^{n} \chi_{Q_{n_l}}-d_i \sum_{l=r_i}^{n} p_l\right\|_p \\
&\quad+\left\| \sum_{k=1}^{i-1} d_k \sum_{l=r_k}^{l_k} \chi_{Q_{n_l}}\right\|_p+\left\| d_i \sum_{l=r_i}^{n} \chi_{Q_{n_l}}\right\|_p \\
&\leqslant 4\ \|R(t)\|_p.
\end{aligned}$$

Note that we put $c_l=0$ for that l which was not used for constructing $P(t)$. In particular, if $l_i<n<r_{i+1}$, then $\sum_{l=N}^{n} c_l p_l=\sum_{l=N}^{l_i} c_l p_l$. ∎

Proof of Theorems 1, 2, *and* 3. Let $\frac{1}{2} \leqslant \sigma_0' < 1$. Now we will use Lemma 3 and induction. Let $f_0 = f$. Let $f_0 = f$, where f is any function from $L_p(a, b)$. Then we find a sequence of step functions $\{R_k\}$, $k \geqslant 1$, numbers

$$N \leqslant N_1 < m_1 < \cdots < N_k < m_k < \cdots,$$

functions f_k, $k \geqslant 1$, and linear combinations $\sum_{l=N_k}^{m_k} c_l p_l$, $k \geqslant 1$, in the system $\{p_l\}$ by induction such that the following is true:

$$f_k = f_{k-1} - \sum_{l=N_k}^{m_k} c_l p_l, \qquad \|f_{k-1} - R_k\|_p < \frac{1}{2^{k+1}}.$$

For each R_k, the linear combination $\sum_{l=N_k}^{m_k} c_l p_l$ is constructed as it was in the proof of Lemma 3. Then we obtain

$$\left\| R_k - \sum_{l=N_k}^{m_k} c_l p_l \right\|_p \leqslant \sigma_0' \, \|R_k\|_p,$$

$$\left\| \sum_{l=N_k}^{n} c_l p_l \right\|_p \leqslant 4 \, \|R_k\|_p, \qquad N_k \leqslant n \leqslant m_k.$$

To prove Theorems 1, 2, 3 we will verify that the series $\sum_{l=1}^{\infty} c_l p_l$ represents f in L_p (we put $c_l = 0$ for the remaining indices l). To finish the proof, let us consider any sufficiently large $n > 0$ and define the index $k \geqslant 1$ such that $N_{k-1} \leqslant n \leqslant m_{k-1}$, $k \geqslant 2$. Then

$$\begin{aligned} \left\| f - \sum_{l=N_1}^{n} c_l p_l \right\|_p &\leqslant \left\| f - \sum_{i=1}^{k-1} \sum_{l=N_i}^{m_i} c_l p_l \right\|_p + \left\| \sum_{l=N_{k-1}}^{n} c_l p_l \right\|_p \\ &\leqslant \|f_{k-2}\|_p + \left\| \sum_{l=N_{k-1}}^{n} c_l p_l \right\|_p \\ &\leqslant \|f_{k-2}\|_p + 4 \, \|R_{k-1}\|_p \leqslant 5 \, \|f_{k-2}\|_p + \frac{4}{2^k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|f_{k-1}\|_p &\leqslant \frac{1}{2^{k-1}} + \sigma_0' \, \|f_{k-2}\|_p \leqslant \frac{1}{2^{k-1}} + \sigma_0' \left(\frac{1}{2^{k-2}} + \sigma_0' \, \|f_{k-3}\|_p \right) \\ &\leqslant \frac{1}{2^{k-1}} + \sigma_0' \frac{1}{2^{k-2}} + (\sigma_0')^2 \frac{1}{2^{k-3}} + \cdots + (\sigma_0')^{k-2} \, \|f\|_p \\ &\leqslant k(\sigma_0')^{k-1} + (\sigma_0')^{k-2} \, \|f\|_p. \end{aligned}$$

It can be easily seen that $\|f_{k-1}\|_p \to 0$ for $k \to \infty$. ∎

CONSEQUENCE 1 [FO, Theorem 1a]. *Let* $\varphi \in L_q(0, 1)$ *for* $1 \leqslant q < \infty$ *and* $\varphi(t)$ *outside* $[0, 1]$ *be considered as equal to zero. If*

$$\int_0^1 \varphi(t)\, dt \neq 0,$$

then $\{\varphi_{n,k}\}$ *is a representation system in* $L_p(0, 1)$ *for any* $0 < p \leqslant q$.

Proof. It is obvious that $\sigma_{n,k} = \sigma < 1$ and $D_{n,k} = [k/2^n, k + 1/2^n]$ for all $n = 0, 1, ..., k = 0, 1, ..., 2^n - 1$.

CONSEQUENCE 2 [FO, Theorem 3]. *Let* $\varphi \in L_1(R)$. *If*

$$\int_R \varphi(t)\, dt \neq 0,$$

then $\{\varphi_{n,k}\}$ *is a representation system in* $L_1(R)$.

CONSEQUENCE 3 [FO, Theorem 1b]. *Let* $\varphi \in L_2(0, 1)$, $\|\varphi\|_2 \neq 0$, *and* φ *outside* $[0, 1]$ *be considered as equal to zero. Then* $\{\varphi_{n,k}\}$ *is a representation system in* $L_p(0, 1)$, $0 < p < 1$.

CONSEQUENCE 4. *Let* $\varphi \in L_2(R) \cap L_p(R)$, $0 < p < 1$, *and* $\|\varphi(t)\|_p \neq 0$. *Then* $\{\varphi_{n,k}\}$ *is a representation system in* $L_p(R)$, $0 < p < 1$.

We will show now that the assumptions in Theorem 1 are important. For brevity, let us consider the case when $(a, b) \equiv (0, 1)$.

(1) First, we will give an example when assumption (5) is violated, but the other assumptions are fulfilled. Let $\inf_n\{d(\varphi_n)\} = \alpha > 0$ and $\text{mes}(\text{supp}\, \varphi_n) \to 0$. Let us, for that case, construct the example of a representation system. Let

$\varphi^1_{n,k}(t) = 1$, $t \in (k/2^n, k + 1/2^n + 1/2^{n+1}) \cup (1 - 1/2^{n+1}, 1)$, $\varphi^1_{n,k}(t) = 0$ at the other points, where $k = 0, 1, ..., 2^{n-1} - 1$, $n = 1, 2, ...$;

$\varphi^2_{n,k}(t) = 1$, $t \in (k + 1/2^n, k + 1/2^n + 1/2^{n+1}) \cup (1 - 1/2^{n+1}, 1)$, $\varphi^2_{n,k}(t) = 0$ at the other points, where $k = 0, 1, ..., 2^{n-1} - 1$, $n = 1, 2, ...$;

$\varphi^1_{n,k}(t) = 1$, $t \in (k/2^n - 1/2^{n+1}, k + 1/2^n) \cup (0, 1/2^{n+1})$, $\varphi^1_{n,k}(t) = 0$ at the other points, where $k = 2^{n-1}, 2^{n-1} + 1, ..., 2^n - 1$, $n = 1, 2, ...$;

$\varphi^2_{n,k}(t) = 1$, $t \in (k/2^n - 1/2^{n+1}, k/2^n) \cup (0, 1/2^{n+1})$, $\varphi^2_{n,k}(t) = 0$ at the other points, where $k = 2^{n-1}, 2^{n-1} + 1, ..., 2^n - 1$, $n = 1, 2....$

It is obvious that the system $\{\varphi^1_{n,k} - \varphi^2_{n,k}\} = \{\psi_{n,k}\}$, $k = 0, 1, ..., 2^n - 1$, $n = 1, 2, ...$, is a system of the following kind: $\{\psi_{n,k}\} = \{\psi(2^n t - k)\}$, $k = 0, ..., 2^n - 1$, $n = 1, 2, ...$, where $\psi(t) = 1$, $t \in [0, 1]$, $\psi(t) = 0$, $t \notin [0, 1]$. Thus the system $\{\psi_l\}_{l=1}^{\infty} = \{\varphi^1_{n,k}, \varphi^2_{n,k}\}$, $k = 0, 1, ..., 2^n - 1$, $n = 1, 2, ...$, is a representation system in $L_p(0, 1)$, $0 < p < \infty$. We can see that for all $f \in L_p[0, 1]$, $0 < p < \infty$, there exists $\sum_{k=1}^{\infty} c_k \psi_k$, $|c_k| \leqslant 1$, such that $\|f - S_{2n}\|_p \to 0$ and $\|f - S_{2n+1}\|_p \to 0$, where $S_{2n} = \sum_{k=1}^{2n} c_k \psi_k$ and $S_{2n+1} = \sum_{k=1}^{2n+1} c_k \psi_k$.

Now we will give an example of a non-complete system. Let $\varphi_{n,k}(t) = 1$, $t \in \bigcup_{i=1}^{\infty} (1/2^i + k/2^{i+n}, 1/2^i + k + 1/2^{i+n})$, $k = 0, 1, ..., 2^n - 1$, $n = 1, 2, ...$; $\varphi_{n,k}(t) = 0$ in all the other points. It is obvious that $\{\varphi_{n,k}(t), k = 0, 1, ..., 2^n - 1, n = 1, 2, ...\} \perp \{\chi_1(2^n t)\}_{n \in Z_+}$, where $\chi_1(t) = 1$, $t \in [0, \frac{1}{2}]$, $\chi_1(t) = -1$, $t \in (\frac{1}{2}, 1]$.

(2) Let now $\int_0^1 \varphi_l(t)\, dt = 0$, $l \geqslant 1$, but the other assumptions, (4), (5), are fulfilled. Let us consider the system $\{\varphi_{n,k}(t)\} = \{\varphi(2^n t - k)\}$, $k = 0, 1, ..., 2^n - 1$, $n = 0, 1, 2, ...$, where $\varphi(t) = f_i(t)$, $t \in (1/2^{i+1}, 1/2^i]$, $i = 0, 1, ...$, and $f_i(t)$ are arbitrary functions such that $\int_{2^{-i-1}}^{2^{-i}} f_i(t)\, dt = 0$, $f_i(t) = 0$, $t \notin (1/2^{i+1}, 1/2^i]$, $i = 0, 1....$ Thus the system $\{\varphi_{n,t}(t), k = 0, 1, ..., 2^n - 1, n = 0, 1, ...\} \perp \{\chi_1(2^n t)\}_{n=0}^{\infty}$.

(3) Let $\sup_n \sigma_n = 1$, and for the system $\{\varphi_{nk}\}$, the conditions (4), (5), and $\int_0^1 \varphi_n(t)\, dt \neq 0$, $n \geqslant 1$, are fulfilled. For this case, let us construct a non-complete system. Since Haar's system $\{\chi_n\}_{n=1}^{\infty}$ is the basis in $L_p(0, 1)$, $p \geqslant 1$ then, by the Theorem of M. Krein, D. Milman, and M. Routman [KrMR] about the stability of a basis, there exists, as a consequence, $\{\delta_n\}_{n=1}^{\infty}$, $\delta_n > 0$, such that for any system $\{g_n\}_{n=1}^{\infty}$, $\|\chi_n - g_n\|_p \leqslant \delta_n$, the system $\{g_n\}_{n=1}^{\infty}$ is a basis in $L_p(0, 1)$, $p \geqslant 1$, too. Then from the systems $\{g_n\}_{n=1}^{\infty}$, for which the condition $\|\chi_n - g_n\|_p \leqslant \delta_n$ is fulfilled, we choose the following system: the following conditions $\|\chi_n - g_n\|_p \leqslant \delta_n$, $\lim_{n \to \infty} \sigma_n = 1$, $\sigma_n < 1$, where

$$\sigma_n = \inf_{\lambda \in R} \left\{ \|(1 - \lambda g_n(t))\, \chi_{D_n}(t)\|_p \frac{1}{|D_n|^{1/p}} \right\},$$

are fulfilled. It is obvious that the system $\{g_n\}_{n=1}^{\infty}$ is a basis, but the system $\{g_n\}_{n=N}^{\infty}$, $N \geqslant 2$, is not a complete one in $L_p(0, 1)$, $p \geqslant 1$, although the conditions (4), (5), and $\int_0^1 \varphi_n(t)\, dt \neq 0$, $n \geqslant 1$, are fulfilled.

(4) Now we give an example of a non-complete system for the case of Theorem 1, when the condition (4) is not fulfilled, but the other conditions are fulfilled. Let $\varphi_n(t) = 1$, $t \in [0, 1/2^n]$, $\varphi_n(t) = 1/2^{2n}$, $t \in (1/2^n, 1]$, $n = 1, 2, ...$ Then $x_n = 0$, $y_n = 1/2^n$, and $d(\varphi_n) = 1/2^n \to 0$, $\sigma_n \leqslant 1/2^n$. We can see that $\{\varphi_n(t)\}_{n \geqslant 1} \perp \{\chi_1(2^n(t + \frac{1}{2}))\}_{n \geqslant 1}$.

(5) Now we will give an example of a non-complete system when the assumption (5) of Theorem 3 is violated, but other assumptions are fulfilled. Let

$\varphi_{2n-1}(t)=1$, $t\in[0,\frac{1}{2})$, $\varphi_{2n-1}(t)=0$, $t\in[\frac{1}{2},1]$, $n=1,2,\ldots$, $\varphi_{2n}(t)=1$, $t\in[\frac{1}{2},1]$;

$\varphi_{2n}(t)=0$, $t\in[0,\frac{1}{2})$, $n=1,2,\ldots$. Then the system $\{\varphi_n\}_{n=1}^{\infty}$ is not a complete system in $L_p(0,1)$, $0<p<1$.

The example from (1) is an example of a representation system in $L_p(0,1)$, $0<p<1$, too.

Remark 1. One easily observes from the proof that Theorem 1 carries over to the spaces $L_1[(a,b)^n]$, or even to L_1 spaces on arbitrary measurable sets $\Omega\subset\mathbf{R}^n$, $n\geqslant 1$.

Remark 2. Obviously, from the proof of Theorem 1, if we take the function $\varphi\in L_1(R^n)$, $n\geqslant 1$, such that $\int_{R^n}\varphi(x)\,dx\neq 0$, then the system

$$\varphi_{k,\mathbf{i}}(\mathbf{x})=\varphi(2^k\cdot\mathbf{x}-\mathbf{i}),\qquad \mathbf{x}\in\mathbf{R}^n,\quad \mathbf{i}\in\mathbf{Z}^n,\quad k\in\mathbf{Z},$$

is a representation system in $L_1(R^n)$, $n\geqslant 1$.

Remark 3. Theorem 1 shows us that the system $\{\varphi_{n,k}\}=\{\varphi(a^n t-bk)\}$, $k=0,1,\ldots,2^n-1$, $n=0,1,\ldots$, where $a>1$, $b>0$, $\varphi(t)=1$, $t\in[0,1]$, $\varphi(t)=0$, at the other points, is an optimal representation system in $L_1(a,b)$ $(\sigma=0)$ in order to quickly converge partial sums with the representative function for the algorithm which is given above. In this context, instead of a number n of elements of the system $\{\varphi_i\}$, one should consider the finite sum $\sum|Q_i|$. For different systems, one should estimate the error of the approach through the finite sum $\sum|Q_i|$.

Remark 4. One easily observes from the proof that Theorem 3 carries over to the spaces $L_p[(a,b)^n]$, $0<p<1$, or even to L_p spaces on arbitrary measurable sets $\Omega\subset\mathbf{R}^n$, $n\geqslant 1$.

Remark 5. Obviously, from the proof of Theorem 3, it is clear that if we take the function $\varphi\in L_p(R^n)\cap L_2(R^n)$, $n\geqslant 1$, $0<p<1$, such that $\|\varphi\|_2\neq 0$, then the system

$$\varphi_{k,\mathbf{i}}(\mathbf{x})=\varphi(2^k\cdot\mathbf{x}-\mathbf{i}),\qquad \mathbf{x}\in\mathbf{R}^n,\quad \mathbf{i}\in\mathbf{Z}^n,\quad k\in\mathbf{Z},$$

is a representation system in $L_p(R^n)$, $0<p<1$, $n\geqslant 1$.

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